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On a Special Form of Annular Surfaces.

By Virgil Snyder.

The equation of a scroll which contains an m-fold linear directrix and another n-fold linear directrix, skew to the first one, is of the form

$$\overset{m}{f}(\lambda, \mu) = 0, \tag{1}$$

wherein

$$\lambda = \frac{a_1x + b_1y + c_1z + d_1w}{a_2x + b_2y + c_2z + d_2w}, \quad \mu = \frac{a_3x + b_3y + c_3z + d_3w}{a_4x + b_4y + c_4z + d_4w};$$

the two directrices being $\lambda = 0$, $\lambda = \infty$; $\mu = 0$, $\mu = \infty$.

The annular surface obtained from this scroll by Lie's equations is of the form

$$\begin{vmatrix} W & 0 & Z & -(X + iY) \\ 0 & W & -(X - iY) & -Z \\ a_1 - \lambda a_2 & b_1 - \lambda b_2 & c_1 - \lambda c_2 & d_1 - \lambda d_2 \\ a_3 - \gamma a_4 & b_3 - \gamma b_4 & c_3 - \gamma c_4 & d_3 - \gamma d_4 \end{vmatrix} = 0, \tag{2}$$

which develops into the form

$$S_1\lambda\mu + S_2\lambda + S_3\mu + S_4 = 0, \tag{3}$$

$S_r = 0$ being the equation of a sphere.

The required surface is the envelope of the sphere (3), whose equation contains the variables λ, μ, connected by the relation (1). It is the tact-invariant of (1), (3) in the λ, μ plane, and contains the variables X, Y, Z, W to the degree $4n(m + n - 1)$, $n \leqq m$. Thus the spherical image of the quadric is a quartic (Dupin's cyclide), and that of the cylindroid is an annular surface of order 8, though for special values of the coefficients $a_r b_r$, $c_r d_r$, it may be made smaller.

By a linear transformation of the directrices, the form of the scrolls in (1) is not essentially changed. The number and reality of the pinch-points are not changed, and the asymptotic lines will be projective. The new forms substituted in (2) will give very different forms in spherical space. If the directrices belong to the complex

$$c \equiv (xw' - x'w) - (yz' - y'z) = 0,$$

their spherical images will be points, and the annular surface will be binodal. The locus of centers will be in the radical plane of these two point-spheres. If the directrices are conjugate polars with regard to c, the surface reduces to a curve traced on the surface of a sphere. If one directrix is the line $z = 0$, $w = 0$, the annular surface becomes a cone or developable, according as the other directrix does or does not belong to c. Finally, if both directrices belong to c and cut (z, w), they become isotropic planes, and the surface becomes one of revolution. The important characteristic of this transformation is that asymptotic lines on the scroll (1) become lines of curvature on the corresponding surface in sphere-space; hence the same scroll, projectively considered, will suffice to define the lines of curvature of cones, a species of developables, surfaces of revolution and annular surfaces having two spherical directrices. The locus of centers in every case lies on a quadric of revolution, which may become a plane.

When the scroll has two coincident directrices, the equation has the form

$$\sum_{r=0}^{n} \phi_{m+n-2r}(\lambda) \, . \, \mu^r = 0, \tag{4}$$

wherein $\phi_s(\lambda)$ is a polynomial in λ of degree s. In this equation λ has the same value as before, but μ is now defined by

$$(a_2x + b_2y + c_2z + d_2w)\,\mu = (a_3x + b_3y + c_3z + d_3w) - \lambda\,(a_4x + b_4y + c_4z + d_4w).$$

The spherical image of this scroll is

$$\begin{vmatrix} W & 0 & Z & -(X + iY) \\ 0 & W & -(X - iY) & -Z \\ a_1 - \lambda a_2 & b_1 - \lambda b_2 & c_1 - \lambda c_2 & d_1 - \lambda d_2 \\ a_3 - \lambda a_4 - \mu a_2 & b_3 - \lambda b_4 - \mu b_2 & c_3 - \lambda c_4 - \mu c_2 & d_3 - \lambda d_4 - \mu d_2 \end{vmatrix} = 0,$$

which develops into

$$\mu = \frac{S_1\lambda^2 + S_2\lambda + S_3}{S_4\lambda + S_5}. \tag{5}$$

The envelope of the sphere (5), when λ, μ are connected by (4), is of degree $4n(m+2n-1)$, $(n \leqq m)$ in X, Y, Z, W.

The spherical image of a scroll defined by an m, $n\,(n \leqq m)$ correspondence in a linear congruence is an annular surface of order $4n(m+n-1)$, when the directrices are skew and of order $4n(m+2n-1)$, when they are coincident. These forms still contain irrelevant factors; by making the directrices satisfy certain conditions, e. g., belong to a given complex, any particular s_i can be eliminated, which reduces the order of the surface. The presence of double generators also reduces the order of the surface, in general, to the form

$$4(mn-\delta),$$

wherein δ is the number of double generators.

Thus, the (2, 2) quartic scroll becomes an annular surface of order 16; it is the transformed surface of the binodal cyclide when the nodes are distinct. The lines of curvature are of order 16 from my article in vol. XXII, p. 96, of the *Journal.* Similarly, the cyclide having three distinct conical points is transformed into a surface of order 12, having lines of curvature of order 8 and one double generator.

Each of these surfaces has two lines of curvature of order half that of the general ones, the curve of contact with the two directrices. Special forms of these are obtained by restricting the pinch-points.

The cyclides which have less than two nodes are not annular surfaces unless the node be biplanar or uniplanar, in which cases they can be readily transformed by the method of equation (5).

An illustration of this method is afforded by Cayley's cubic scroll, which can be written in the form

$$xy^2 = (x+y)(xz+yw),$$

or

$$\lambda^2 = (1+\lambda)\mu. \tag{6}$$

This is the degraded Kummer surface [(13) 2],* as singular surface of a quadratic complex.

Equation (5) becomes

$$\mu(X^2+Y^2+Z^2) - \lambda XW + (\lambda^2+\lambda)ZW = 0,$$

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* A. Weiler, Ueber die verschiedenen Gattungen der Complexe zweiten Grades. Mathematische Annalen, vol. 7. See pp. 205–6.

and the envelope of this sphere, subject to the condition (6), is

$$(X^2 + Y^2 + Z^2)^2 + 2W(2Z - X)(X^2 + Y^2 + Z^2) + W^2X^2 = 0,$$

which is a cyclide having a uniplanar point at the origin, and is the type [(13) 1] of Loria.† Three lines of the surface pass through the origin and lie in $X = 0$.

The lines of curvature are defined by the intersection with the surface of a tangent pencil of spheres, whose point of contact is the origin ($W \neq 0$), and the common tangent plane is $Y = 0$. They are of order 4. By the transformation in question, this surface becomes an annular surface of order 12, and whose lines of curvature are defined by the intersection with the surface of a pencil of spheres having for circle of intersection (trajectory circle of a special spherical congruence) a line along which the double directrix touches the surface. The other lines of curvature are of order 8.

The relation between the scroll and the annular surface may be made clearer by giving more of the details of the transformation.

From every point of the double directrix (x, y) issues one real generator of the scroll; there are no pinch-points, but for one point on the directrix the generator which issues from that point coincides with the directrix itself.

The director becomes a sphere S, its points become one generation of minimum lines upon it, and the planes through (x, y) projective with points upon it, go into the second generation of S, projective with the first. Passing through each point P of (x, y) on the scroll and lying in its associated plane π through the director, is a generator g. Through the point on S, determined by the minimum lines m_P and m_π, is a sphere G, containing both minimum lines (touching S). The locus of the points of contact is the trajectory circle of S. Upon this circle are no points at which all the lines of curvature touch each other, but at one of them the generating sphere coincides with S. The generating sphere is real for every point of the trajectory circle.

The locus of centers of the generating spheres (one mantle of the curve of centers) is a curve passing through the center of S and lying on the cone of revolution determined by this point and the trajectory circle. The axis of this cone is the curve of centers of the other system. The only plane line of curvature is the plane of the trajectory circle.

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† G. Loria, Recherche intorno alla geometria della sfera. . . . Memorie della Acc. Reale di Torino, vol. 36, ser. II, 1884. See p. 100. These types have been more minutely classified by Bôcher in his Reihen entwickelung. . . . Leipzig, 1894.

There is no distinct cyclide which corresponds to the $\overline{(3,\,1)}$ scroll. In the equation of the cyclide there are 13 constants, while the equation of the Kummer surface contains 18; hence, several distinct types of general annular surfaces are confounded when the points of space are chosen to represent one fundamental complex.

When the cubic scroll is given in the form

$$y^3 + x(zx + yw) = 0,$$

the corresponding cyclide is of the third order and has a uniplanar point. Its equation is

$$4(X^2 + Y^2 + Z^2)(Z - X) = Z^2.$$

Besides the minimum lines $X \pm iY = 0$, $Z = 0$, the real line $X = 0$, $Z = 0$ lies on the surface and passes through the uniplanar point.

Although this is a contact transformation and transforms spheres into spheres such that lines of curvature are covariant, yet the pencil of tangent spheres which defines the lines of curvature on the cyclide does not transform into the pencil of spheres which defines the lines of curvature of the new surface; in fact, the tangent pencil into which the old tangent pencil is transformed plays no part in the transformed surface.*

On p. 253 of volume XXII of the *Journal*, I gave the equation of a [3, 1] scroll whose asymptotic lines are of order three. The spherical image of this surface is of peculiar type. Let

$$\mu = \frac{1 - 3\lambda^2}{3\lambda - \lambda^3}, \quad \lambda = \frac{x}{y}, \quad \mu = \frac{z}{w};$$

the line $x = 0$, $y = 0$ belongs to c, hence, it will be a point; the line $z = 0$, $w = 0$ goes into the plane at infinity, hence, all the generating spheres of the surface become planes which pass through a fixed point, i. e., the surface is a cone. The equation of this cone is found to be

$$(4x^2 + y^2 + 4z^2)^3 - 27z^2y^4 = 0.$$

Its lines of curvature are cut from the surface by the pencil of spheres concentric with its vertex,

$$x^2 + y^2 + z^2 = k^2w^2.$$

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*The remark made by Casey, Phil. Trans., vol. 161, p. 627, that all the lines of curvature of a trinodal cyclide are circles is incorrect. This is only true of a limiting case of Dupin's cyclide, in which two nodes coincide. It is type [(21)(11)] of Loria. In my classification of Dupin cyclides I called this a "pinch" cyclide.

By eliminating z (e. g.) between these two equations, the projection of the lines of curvature on the x, y plane becomes

$$[kw(8k^2 - 9y^2) \pm 3\sqrt{3}xy^2][kw(8k^2 - 9y^2) - 3\sqrt{3}xy^2] = 0.$$

Both surfaces are symmetric about all the axes, hence each curve is of order 6.

By transforming the cone by the above method, keeping the directrices in the point-complex, a binodal annular surface is obtained whose order is 12, and all of its lines of curvature are still of order 6.

If, instead of x, y and z, w, any two lines which cut the latter and which do not belong to c had been chosen for directrices, the spherical image would be an annular surface of order 12 with two plane directors; hence, all of its lines of curvature are of order 6, and are plane curves. The surface is not a surface of revolution. The two sextics which are cut from the surface by these planes have 36 points of intersection; of these, 18 lie in the circle-points of the cutting plane and six lie on the axis of the planes; the other points are either nodes on the surface or points of tangency of the cutting plane. The latter alternative is excluded because it contains a line of curvature and consequently cuts the surface everywhere at the same angle (Joachimstal's theorem). Hence, the surface has a nodal line of order 12. This surface is thus seen to have a remarkable analogy to Dupin's cyclide. The latter is an annular surface of order 4, and all of its lines of curvature of the second system are plane curves, which factor into conics through the circle-points (circles) and cut each other in points on the axis of the two planes. These curves have only four points in common, two at the circle-points and two on the axis; hence Dupin's cyclide has no other nodal line. This transformation is more general than that by reciprocal radii, which it includes as a particular case. It is not, in general, a point transformation, but, as was shown by Lie, it can be expressed in terms of motion, reciprocation and Bonnet's dilatation, the last transforming a point into a sphere. Analytically, it is expressed as the most general linear transformation of the hexaspherical coordinates which leaves the quadratic form $\Pi = 0$ invariant.*

By means of it a whole class of surfaces can be derived which have the same property as that just cited. Let $f(x, y) = 0$ be the equation of any plane curve.

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*Cf. A. Loewy, "Ueber die Transformationen einer quadratischen Form in sich selbst mit Anwendung auf Linien- und Kugelgeometrie." Nova Acta, Leopoldina, vol. 65, pp. 1–66. It is unfortunate that the word dilatation is used in so different senses. Loewy calls this "eine von Ossian Bonnet angegebene reciproke Umformung."

If it be revolved about any line in the plane, it will generate a surface of revolution whose order is, in general, twice that of the curve, and all of whose lines of curvature are plane curves. If, now, this surface be transformed into line space, it will become a scroll contained in a linear congruence having asymptotic lines of order just half that of the general surface of this type. If the directrices of this congruence be transformed into any lines which cut $z = 0$, $w = 0$ but do not belong to c, the characteristic property of the surface remains unchanged. Now let this new surface be transformed back into spherical space. It will be an annular surface of order higher than that of the surface of revolution, unless the curve $f = 0$ has each circle point of its plane for a multiple point of order half that of f. The new annular surface will not be a surface of revolution, but all of its lines of curvature will be plane, and the curve of each section by these planes will break up into two curves. The nodal line is not the transformed curve of the circles, loci of nodes in f. The planes belong to the same axial pencil, the axis being always a finite line. The planes of the circles of the other system of curvature all touch a cylinder whose generators are perpendicular to this line.

CORNELL UNIVERSITY, *May* 29, 1900.